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Analysis II (Part II of Calculus) Academic year 2023/2024.

# Final Exam Duration: 1h30 mn.
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**Exercise 1.** *:(7 Pts)*

1) *Give the Taylor's formula with the reminder of Lagrange to the order* $n = 3$ *at* $x_0 = 0$ *for* $f(x) = e^{-3x}$.

2) *Give the limited development of* $f(x) = \sqrt{x}$ *to the order* $n = 3$ *at* $x_0 = \frac{1}{2}$.

3) *Calculate using L.D, the following limit*

$$\lim_{x \longrightarrow 0} \frac{e^x - \sqrt{1+x}}{2\arctan(x) - \arcsin(x)}.$$

**Exercise 2.** *(8 Pts)*

*Let* $M = \begin{pmatrix} 2 & 1 & -2 \\ -1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$ *and* $N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

1. *Compute the determinant of* $M$ *and* $N$*, then* deduce $\det(3M)$, $\det(MN)$ *and* $\det((M^t)N)$.
2. *Compute the inverse* $M^{-1}$ *of* $M$ *using the co-factor matrix.*
3. *Deduce the solution of the following linear system :* $\begin{cases} 2\alpha + 1\beta - 2\gamma = 1 \\ -\alpha + 2\beta + 3\gamma = 0 \\ \alpha + 2\beta + \gamma = -1 \end{cases}$

**Exercise 3.** *(5 Pts)*

1. *Using integration by parts compute* $\int e^x \sin x dx$.
2. *Compute the following primitives*
   (a) $\int \frac{\sin x}{\cos^2 x + 4} dx$.
   (b) $\int \frac{6}{x^2 + 5x + 4} dx$.

***Indication (LD at*** $x_0 = 0$ ***to the order*** $n$***):***

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - ...(-1)^n \frac{x^{2n+1}}{2n+1} + o(x^{2n+2}),$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + ... + (-1)^n \frac{2n!}{2^{2n+1} n!(n+1)!} x^{n+1} + o(x^{n+1}).$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + o(x^{n+1}).$$

$$\arcsin x = x + \frac{1 \cdot x^3}{2 \cdot 3} + \frac{1 \cdot 3x^5}{2 \cdot 4 \cdot 6} + ... + \frac{1 \cdot 3 \cdot 5...(2n-1)x^{2n+1}}{2.4.6...(2n)(2n+1)} + o(x^{2n+2}).$$

# Final Exam solution.

1. The Taylor's formula with the reminder of Lagrange to the order $n = 3$ at $x_0 = 0$ for $f(x) = e^{-3x}$ is given by:
$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(c)}{4!}x^4,$
such that $0 < c < x$ or $x < c < 0$ ·························· **(0.5+0.25 pt)**

As the function $f$ belongs to $C^\infty(\mathbb{R})$ ···································· **(0.25 pt)**

$\forall x \in \mathbb{R}:\ f'(x) = -3e^{-3x},\ f^{(2)}(x) = 9e^{-3x},\ f^{(3)}(x) = -27e^{-3x},\ f^{(4)}(x) = 81e^{-3x}$ **(0.5pt)**

Then, $f'(0) = -3,\ f^{(2)}(0) = 9,\ f^{(3)}(0) = -27,\ f^{(4)}(c) = 81e^{-3c}$ ········· **(0.5 pt)**
Hence,
$f(x) = 1 + -3x + \frac{9}{2!}x^2 - \frac{27}{6}x^3 + \frac{81e^c}{24}x^4,$ with $0 < x < c$ or $x < c < 0$ ··· **(0.5 pt)**

2. Give the limited development of $f(x) = \sqrt{x}$ to the order $n = 3$ at $x_0 = \frac{1}{2}$.
We put $t = x - \frac{1}{2}$ then $x = t + \frac{1}{2}$ ···································· **(0.25 pt)**
$x \to \frac{1}{2}$ then $t \to 0$ ············································· **(0.25 pt)**
Then,
$f(x) = f(t + \frac{1}{2}) = \sqrt{t + \frac{1}{2}} = \frac{\sqrt{2}}{2}\sqrt{1 + 2t}$ ························· **(0.5 pt)**
$t \to 0$ then $2t \to 0$ ············································· **(0.25pt)**
Using indication (LD at point 0).

$$\sqrt{1+2t} = 1 + \frac{1}{2}(2t) - \frac{1}{8}(2t)^2 + \frac{1}{16}(2t)^3 + o(t^3) \quad \cdots\cdots \textbf{(0.25 pt)}$$
$$= 1 + t - \frac{1}{2}t^2 + \frac{1}{2}t^3 + o(t^3) \quad \cdots\cdots \textbf{(0.25 pt)}$$

multiplying by $\frac{\sqrt{2}}{2}$ and replacing $t$ by its value, gives

$$f(x) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{1}{2}) - \frac{\sqrt{2}}{4}(x - \frac{1}{2})^2 + \frac{\sqrt{2}}{4}(x - \frac{1}{2})^3 + o((x - \frac{1}{2})^3) \quad \cdots\cdots \textbf{(0.25 pt)}$$

3. We use indication (LD at $x_0 = 0$ to the order $n = 1$)
$e^x = 1 + x + o(x),\ \sqrt{(1+x)} = 1 + \frac{1}{2}x + o(x)$ ··········· **(0.25 + 0.25 pt)**
$2\arctan(x) = 2x + o(x),\ \arcsin(x) = x + o(x)$ ··········· **(0.25 + 0.25 pt)**
Hence the limit

$$\lim_{x \longrightarrow 0} \frac{e^x - \sqrt{1+x}}{2\arctan(x) - \arcsin(x)} = \lim_{x \longrightarrow 0} \frac{1 + x + o(x) - 1 - \frac{1}{2}x - o(x)}{2x + o(x) - x + o(x)} = \lim_{x \longrightarrow 0} \frac{+\frac{1}{2}x + o(x)}{x + o(x)} \cdots \textbf{(0.5}pt\textbf{)}$$

$$= \lim_{x \longrightarrow 0} \frac{\frac{1}{2}x + x.\varepsilon_1(x)}{x + x.\varepsilon_2(x)} = \lim_{x \longrightarrow 0} \frac{x\left(\frac{1}{2} + \varepsilon_1(x)\right)}{x\,(1 + \varepsilon_2(x))} = \frac{1}{2}.\textbf{(0.25+0.25+0.25 pt)}$$

$\Big($because $\lim_{x \longrightarrow 0} \varepsilon_1(x) = 0 \quad \lim_{x \longrightarrow 0} \varepsilon_2(x) = 0\Big)$ ···················· **(0.25pt)**

1. • $\det(M) = 2\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix}$ ........................ (**0.5 pt**)

$= 2(2-6) - (-1-3) - 2(-2-2) = 4$ ........................ (**0.5 pt**)

• $\det(N) = 1 \cdot (-2) \cdot 3 = -6$. (because $N$ is triangular matrix) ...... (**0.25+0.25 pt**)
**Further, according to the stated properties during the course, we have:**

• $\det(3M) = 3^3 \det(M) = 3^3 \cdot 4 = 108$ ........................ (**0.25+0.25 pt**)

• $\det(MN) = \det(M) \cdot \det(N) = 4 \cdot (-6) = -24$ ............ (**0.25+0.25 pt**)

• $\det(M^t N) = \det(M^t) \cdot \det(N) = \det(M) \cdot \det(N) = 4 \cdot (-6) = -24$ ... (**0.5 pt**)

2. **The inverse of $M$: As $\det(M) \neq 0$ the matrix $M$ admits an inverse** ... (**0.5pt**)

$M^{-1} = \dfrac{1}{\det(M)} (\mathrm{Com}(M))^t$ ........................................ (**0.25 pt**)

$$\mathrm{Com}(M) = \begin{pmatrix} +\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} -1 & 3 \\ 1 & 1 \end{vmatrix} & +\begin{vmatrix} -1 & 2 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} \end{pmatrix} \cdots\cdots\cdots\cdots (\mathbf{0.25 \times 9\ pt})$$

which gives us

$\mathrm{Com}(M) = \begin{pmatrix} -4 & 4 & -4 \\ -5 & 4 & -3 \\ 7 & -4 & 5 \end{pmatrix} \cdots (\mathbf{0.25}\ pt); (\mathrm{Com}\, M)^t = \begin{pmatrix} -4 & -5 & 7 \\ 4 & 4 & -4 \\ -4 & -3 & 5 \end{pmatrix} \cdots (\mathbf{0.25}\ pt)$

To conclude that

$$M^{-1} = \frac{1}{4}\begin{pmatrix} -4 & -5 & 7 \\ 4 & 4 & -4 \\ -4 & -3 & 5 \end{pmatrix} = \begin{pmatrix} -1 & \frac{-5}{4} & \frac{7}{4} \\ 1 & 1 & -1 \\ -1 & \frac{-3}{4} & \frac{5}{4} \end{pmatrix} \cdots\cdots\cdots\cdots (\mathbf{0.25\ pt})$$

3. **Deduce the solution of the following linear system :**

$$\begin{cases} 2\alpha + 1\beta - 2\gamma = 1 \\ -\alpha + 2\beta + 3\gamma = 0 \\ \alpha + 2\beta + \gamma = -1 \end{cases} \cdots\cdots\cdots (I)$$

$$(I) \iff M \cdot X = b \quad \text{such that} X = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad \text{and } b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdots (\mathbf{0.25}pt)$$

$\iff M^{-1} \cdot M \cdot X = M^{-1} \cdot b$ .................................... (**0.25 pt**)

$\iff X = M^{-1} \cdot b \quad \big(\text{because } M^{-1} \text{ exist and } M^{-1} \cdot M = I_3\big) \cdots (\mathbf{0.25}pt)$

The unique solution of $(I)$ is

$$\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix} = M^{-1}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix} = \begin{pmatrix}-1 & \frac{-5}{4} & \frac{7}{4}\\ 1 & 1 & -1\\ -1 & \frac{-3}{4} & \frac{5}{4}\end{pmatrix}\begin{pmatrix}1\\ 0\\ -1\end{pmatrix} = \begin{pmatrix}-\frac{11}{4}\\ 2\\ -\frac{9}{4}\end{pmatrix} \cdots (\mathbf{0.25 + 0.25\ pt})$$

1. **Compute** $I = \int e^x \sin x dx$.
   We will integrate by parts twice in a row with the usual formula

$$\int fg'dx = fg - \int f'gdx \cdots\cdots\cdots\cdots\cdots\cdots (\mathbf{0.25\ pt})$$

We put $f = \sin(x)$ and $g' = e^x \cdots (\mathbf{0.25\ pt})$ then $f' = \cos(x)$, and $g = e^x \cdots (\mathbf{0.25\ pt})$
Hence,

$$I = e^x \sin(x) - \int e^x \cos(x)dx \cdots\cdots\cdots\cdots\cdots\cdots (\mathbf{0.25\ pt})$$

The second one is done as follows:
we put $f = cos(x)$, $g' = e^x \cdots (\mathbf{0.25\ pt})$ that is $f' = -\sin(x)$ and $g = e^x \cdots (\mathbf{0.25\ pt})$
which gives us

$$\begin{aligned} I &= e^x \sin(x) - \left(e^x \cos(x) - \int -e^x \sin(x)dx\right) \cdots\cdots\cdots\cdots (\mathbf{0.25\ pt}) \\ &= e^x \sin(x) - \left(e^x \cos(x) + \int e^x \sin(x)dx\right) \cdots\cdots\cdots\cdots (\mathbf{0.25\ pt}) \\ &= \frac{1}{2}e^x\left(\sin(x) - \cos(x)\right) + C. \qquad C \in \mathbb{R} \cdots\cdots\cdots\cdots (\mathbf{0.25\ pt}) \end{aligned}$$

2. (a) **Evaluate** $I = \int \dfrac{\sin x}{\cos^2 x + 4}dx$.
   Substitute $t = \dfrac{\cos(x)}{2}$ which gives us $dt = -\dfrac{\sin(x)}{2}dx \cdots\cdots\cdots (\mathbf{0.25\ pt})$

$$\begin{aligned} I &= -\int 2\frac{1}{4t^2+4}dt = -\frac{1}{2}\int \frac{1}{t^2+1}dt \cdots\cdots (\mathbf{0.25 + 0.25\ pt}) \\ &= -\frac{1}{2}\arctan(t) + C. \qquad C \in \mathbb{R} \cdots (\mathbf{0.25\ pt}) \end{aligned}$$

   (b) **Calculate** $I = \int \dfrac{6}{x^2+5x+4}dx$.
   As the discriminant of the polynomial in the denominator $(\Delta = 9)$ is positive, we can factorize the denominator and write $\cdots\cdots\cdots\cdots (\mathbf{0.25\ pt})$

$$\begin{aligned} I &= 6\int \frac{1}{(x+1)(x+4)}dx = 6\int \left(\frac{\frac{1}{3}}{(x+1)} + \frac{\frac{-1}{3}}{(x+4)}\right)dx. (\mathbf{0.25 + 0.25 + 0.25\ pt}) \\ &= 2\left(\int \frac{1}{x+1}dx - \int \frac{1}{x+4}dx\right) \cdots\cdots\cdots\cdots (\mathbf{0.25\ pt}) \\ &= 2\ln(|x+1|) - 2\ln(|x+4|) + C. \qquad C \in \mathbb{R} \cdots\cdots\cdots (\mathbf{0.25\ pt}) \end{aligned}$$